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A NON-STATIONARY DYNAMICAL PERIODIC CONTACT PROBLEM FOR A HOMOGENEOUS ELASTIC HALF-PLANE*

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The problem of determining the contact stresses under a periodic system of stamps located on the boundary of a homogeneous elastic half-plane and moving under the effect of a load, identical for all stamps, that is arbitrary in time, is investigated. The problem reduces to solving a Fredholm integral equation of the first kind for the Laplace transform of the contact stresses. The stresses are sought in the form of a double expansion in Chebyshev polynomials of the linear coordinate and Laguerre polynomials of time. The coefficients of the expansions are determined recursively from an infinite quasiregular system of linear algebraic equations.

Despite the fact that the static periodic contact problems of the theory of elasticity, on the one hand (/1-6/, say), and dynamic problems for a finite number if stamps on the other (see the survey in /17/), have been studied repeatedly by different investigators, so far as we know, the plane non-stationary dynamical periodic contact problem has still not been examined at all.

1. A system of vertical unit impulses at the points

$$x = ml\ (m = 0, \pm 1, \pm 2, \ldots) \quad p(x, t) = \sum_{m=-\infty}^{\infty} \delta(x - ml)\,\delta(t) \tag{1.1}$$

where ($\delta(t)$ is the delta function), is applied to the boundary of a homogeneous elastic half-plane. The Ox axis is directed along the half-plane boundary. The variable x and the time t are assumed to be dimensionless; the length scale is a and the time scale is a/c_2. Here a is a certain parameter with the dimensions of length, and c_2 is the transverse velocity of wave propagation in an elastic half-space.

Substituting (1.1) into (1.24) in /7/ and using the equation

$$\sum_{m=-\infty}^{\infty} e^{iml\xi} = \frac{2\pi}{l} \sum_{k=-\infty}^{\infty} \delta\left(\xi - k\frac{2\pi}{l}\right)$$

we obtain a function $\bar{v}_1(x, s)$ that is the Laplace transform of the vertical displacement of a boundary point of the half-plane with abscissa x due to the action of a periodic system of concentrated unit impulses

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$$\bar v_1(x,s)=\frac{aT}{2\pi\mu}\left[\frac{\beta}{s}+2\sum_{k=1}^{\infty}\bar F(kT,s)\cos kTx\right],\qquad(1.2)$$
$$\bar F(\xi,s)=\frac{s^2\sqrt{\xi^2+\beta^2s^2}}{R(\xi,s)}$$
$$R(\xi,s)=(2\xi^2+s^2)-4\xi^2\sqrt{\xi^2+\beta^2s^2}\sqrt{\xi^2+s^2},\quad \beta=c_2/c_1,\quad T=2\pi/l$$

where c_1 and c_2 are the longitudinal and transverse wave velocities in the elastic half-space, and μ is Lamé's constant. To extract the single-valued branches of the radicals $\sqrt{\xi^2+\beta^2s^2}$ and $\sqrt{\xi^2+s^2}$ for fixed real ξ, slits are made in the s plane from the branch point $s=\pm i\xi/\beta$, $s=\pm i\xi$ to infinity along the imaginary half-axis on which these points are located. We select those branches of the radicals which take positive values on the real axis.

The bar in (1.2) and hereafter denotes the Laplace transform.

2. Consider a system of flat stamps, periodic with period l, that lie on the boundary of a homogeneous elastic half-plane (Fig.1). It is assumed that there is no friction between the stamps and the half-plane and that the length of the contact area is constant and equal to the width of the stamp.

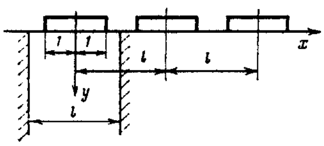

Fig.1

Note that in dimensional units, the length of the stamp width is taken at $2a$, therefore, it equals 2 in dimensionless units.

Let $P_{0r}(t)$ and $M_{0r}(t)$ be the principal vector and principal moment of the foundation reactive forces acting on each of the stamps

$$P_{0r}(t)=a\int_{-1}^{1}p(x,t)\,dx,\quad M_{0r}(t)=a^2\int_{-1}^{1}xp(x,t)\,dx\qquad(2.1)$$

Let us form the stamp equation of motion

$$m_0'\frac{c_2^2}{a^2}\frac{d^2w'}{dt^2}=P_0(t)-P_{0r}(t),\quad J_0'\frac{C_2^2}{a^2}\frac{d^2\varphi}{dt^2}=M_0(t)-M_{0r}(t)\qquad(2.2)$$

Here m_0', J_0' is the stamp mass and axial moment of inertia, and w' and φ are its vertical translational and angular displacement.

Applying a Laplace transform to (2.1) and (2.2), and assuming the angular displacement to be small, we will have

$$m_0s^2\bar w_0'(s)=\bar P_0(s)-\bar P_{0r}(s),\quad J_0s^2\bar w''(x,s)=[\bar M_0(s)-\bar M_{0r}(s)]x\qquad(2.3)$$
$$\bar P_{0r}(s)=a\int_{-1}^{1}\bar p(x,s)\,dx,\quad \bar M_{0r}(s)=a^2\int_{-1}^{1}x\bar p(x,s)\,dx\qquad(2.4)$$
$$m_0=m_0'c_2^2/a^2,\quad J_0=J_0'C_2^2/a^2$$

Here $w_0''(x,s)$ denotes the displacement of the points of the lower faces of the stamp because of its rotation. It is assumed that at the initial instant

$$w_0'(0)=\frac{\partial w_0'(0)}{\partial t}=0,\quad w_0''(x,0)=\frac{\partial w''(x,0)}{\partial t}=0$$

In combination with the periodicity of the system of stamps and loads, these conditions ensure the periodicity and boundedness of the stamp displacements on the whole half-plane boundary at each instant of time.

The Laplace transforms of the displacements of points of the stamp base are determined from the formula

$$\bar w_0(x,s)=\bar w_0'(s)+\bar w_0''(x,s)=\frac{\bar P_0(s)-\bar P_{0r}(s)}{m_0s^2}+\frac{\bar M_0(s)-\bar M_{0r}(s)}{J_0s^2}\qquad(2.5)$$

On the other hand, the Laplace transform of the function $v(x,t)$, which describes the veritical displacements of the half-plane boundary points under the effect of the loads $p(x,t)$, that are periodic in x, has the following form:

$$\bar v(x,s)=\int_{-1}^{1}\bar v_1(x-\xi,s)\,\bar p_0(\xi,s)\,d\xi\qquad(2.6)$$

Equating the right sides of (2.5) and (2.6), we arrive at the integral equation

$$\int_{-1}^{1}\bar v_1(x-\xi,s)\,\bar p_0(\xi,s)\,d\xi=\frac{\bar P_0(s)-\bar P_{0r}(s)}{m_0s^2}+\frac{\bar M_0(s)-\bar M_{0r}(s)}{J_0s^2}x\quad(|x|\leqslant1)\qquad(2.7)$$

3. We represent the solution of (2.7) in the form of the sum

$$\bar{p}_0(\xi, s) = \bar{p}_{00}(\xi, s) + \bar{p}_{01}(\xi, s)$$

where $\bar{p}_{00}(\xi, s)$, $\bar{p}_{01}(\xi, s)$ are solutions of (3.1) and (3.2), respectively

$$\int_{-1}^{1} \bar{v}_1(x-\xi, s)\,\bar{p}_{00}(\xi, s)\,d\xi = \frac{\bar{P}_0(s) - \bar{P}_{0\tau}(s)}{m_0 s^2} \quad (|x| \leqslant 1) \tag{3.1}$$

$$\int_{-1}^{1} \bar{v}_1(x-\xi, s)\,\bar{p}_{01}(\xi, s)\,d\xi = \frac{\bar{M}_0(s) - \bar{M}_{0\tau}(s)}{J_0 s^2} \quad (|x| \leqslant 1) \tag{3.2}$$

We seek the solution of the integral equation (3.1) in the form of a series in even Chebyshev polynomials

$$\bar{p}_{00}(\xi, s) = \sum_{j=0}^{\infty} \bar{A}_{2j}(s) \frac{T_{2j}(\xi)}{\sqrt{1-\xi^2}} \tag{3.3}$$

which corresponds to determining the contact stresses from the formula

$$p_{00}(\xi, t) = \sum_{j=0}^{\infty} A_{2j}(t) \frac{T_{2j}(\xi)}{\sqrt{1-\xi^2}} \tag{3.4}$$

where $A_{2j}(t)$ are unknown functions of time.

We substitute (3.3) into (3.1) and (2.4), then multiply (3.1) by $T_{2n}(x)/\sqrt{1-x^2}$ and integrate between -1 and +1 with respect to x. Using the equations ($J_{2n}(x)$ is the Bessel function of the first kind)

$$\int_{-1}^{1} \frac{T_0(x)}{\sqrt{1-x^2}}\,dx = \pi, \quad \int_{-1}^{1} \frac{T_{2n}(x)}{\sqrt{1-x^2}} \cos kTx\,dx = (-1)^n \pi J_{2n}(kT)$$
$$(n = 0, 1, 2, \ldots)$$

we will have

$$\bar{P}_{0\tau}(s) = a\pi \bar{A}_0(s) \tag{3.5}$$

$$\sum_{j=0}^{\infty} B_{0,2j}(s)\,\bar{A}_{2j}(s) = g\,\frac{\bar{P}_0(s)}{s^2 \pi a} \tag{3.6}$$

$$\sum_{j=0}^{\infty} B_{2n,2j}(s)\,\bar{A}_{2j}(s) = 0 \quad (n = 1, 2, 3, \ldots)$$

$$B_{00}(s) = \frac{\beta}{s} + \frac{g}{s^2} + 2\sum_{k=1}^{\infty} \bar{F}(kT, s)\,J_0^2(kT), \quad g = \frac{2\mu\pi}{m_0 T} \tag{3.7}$$

$$B_{2n,2j}(s) = 2(-1)^j \sum_{k=1}^{\infty} \bar{F}(kT, s)\,J_{2n}(kT)\,J_{2j}(kT) \quad (j, n = 1, 2, 3, \ldots)$$

4. We make the change of parameter $s = 1/p$ in the infinite system of linear algebraic equations (3.6) that depend on the complex parameter s. Under this transformation, the half-plane $\mathrm{Re}\, s > 1/2$ is mapped on the interior of the unit circle $|p-1| < 1$. As follows from the second formula in (1.2), the function $\bar{F}(kT, s)$ is analytic in the right half-plane, and $\bar{F}(kT, s) \sim \beta/s$ as $s \to \infty$, $\mathrm{Re}\, s > \gamma$ for any $\gamma > 0$. Hence it follows that the function $\bar{F}(kT, 1/p)$ is analytic in the unit circle mentioned and is expanded there in the following series:

$$\bar{F}\left(kT, \frac{1}{p}\right) = p \sum_{m=0}^{\infty} F_{km}(p-1)^m \tag{4.1}$$

Then

$$B_{2n,2j}\left(\frac{1}{p}\right) = p \sum_{m=0}^{\infty} B_{2n,2j,m}(p-1)^m \tag{4.2}$$

$$B_{2n,2j,m} = \begin{cases} \beta + 2C_{000} \; (n = j = m = 0) \\ g + 2C_{001} \; (n = j = 0, m = 1) \\ 2(-1)^j C_{2n,2j,m} \; \text{(for the remaining values of } n, j, m) \end{cases} \tag{4.3}$$

$$C_{2n,2j,m} = \sum_{k=1}^{\infty} F_{km} J_{2n}(kT)\,J_{2j}(kT) \tag{4.4}$$

The following system representation results from (4.2)

$$\sum_{j=0}^{\infty} A_{2j}\left(\frac{1}{p}\right) \sum_{m=0}^{\infty} B_{2n,2j,m}(p-1)^m = pD_{2n}(p) \quad (n=0,1,2,\ldots) \tag{4.5}$$

$$D_0(p) = \frac{g}{\pi a} P_0\left(\frac{1}{p}\right), \quad D_{2n}(p) = 0 \quad (n>0)$$

We shall seek the function $A_{2j}(1/p)$ in the form of the follwing series:

$$A_{2j}\left(\frac{1}{p}\right) = p\sum_{m=0}^{\infty} A_{2j,m}(p-1)^m \quad (j=0,1,2,\ldots) \tag{4.6}$$

Expanding the function

$$P_0\left(\frac{1}{p}\right) = \sum_{m=0}^{\infty} Q_m (p-1)^m \tag{4.7}$$

in series, substituting (4.6) and (4.7) into (4.5), and equating the coefficients of identical powers $p-1$, we arrive at a recursion formula for the infinite systems of linear algebraic equations in the coefficients $A_{2j,m}$ $(j, m = 0, 1, 2, \ldots)$

$$B_0 a_m = e_m \quad (m=0,1,2,\ldots) \tag{4.8}$$

$$e_0 = d_0, \; e_m = d_m - \sum_{k=0}^{m-1} B_{m-k} a_k \quad (m=1,2,3,\ldots) \tag{4.9}$$

$$B_m = \| B_{2n,2j,m} \|_{j,\,n=0}^{\infty}, \quad a_m = \| A_{2n,m} \|_{n=0}^{\infty}$$

$$d_m = \| D_{2n,m} \|_{n=0}^{\infty}; \quad D_{0m} = \frac{g}{\pi a} Q_m, \quad D_{2n,m} = 0$$

$$(m=0,1,2,\ldots; \; n=1,2,3,\ldots)$$

Series (4.4) converges.

To prove the convergence of series (4.4), we will find the original of the function $\bar{F}(kT,s)$. Evaluating the appropriate integral by contour integration, we will have /7/

$$F(kT,t) = \psi_1(\zeta_1)\sin(kT\zeta_1 t) + \frac{2}{\pi}\int_1^{\infty} \psi(\zeta)\sin(kT\zeta t)\,d\zeta \tag{4.10}$$

$$\psi_1(\zeta) = -\frac{\zeta\sqrt{1-\beta^2\zeta^2}}{2(2-\zeta^2-\beta^2 Z - 1/Z)}, \qquad Z = \frac{\sqrt{1-\zeta^2}}{\sqrt{1-\beta^2\zeta^2}}$$

$$\psi(\zeta) = \begin{cases} \dfrac{4\zeta^2(1-\beta^2\zeta^2)\sqrt{\zeta^2-1}}{(2-\zeta^2)^4 + 16(1-\beta^2\zeta^2)(\zeta^2-1)} & \left(1 \leqslant \zeta \leqslant \dfrac{1}{\beta}\right) \\[2ex] \dfrac{\zeta^2\sqrt{\beta^2\zeta^2-1}}{(2-\zeta^2)^2 + 4\sqrt{\beta^2\zeta^2-1}\sqrt{\zeta^2-1}} & \left(\dfrac{1}{\beta} \leqslant \zeta < \infty\right) \end{cases}$$

The number $\zeta_1 > 0$ is determined by the location of the poles of the function $R(1,s)$ at the points $s = \pm i\zeta_1$.

We apply a Laplace transform to (4.10)

$$\bar{F}(kT,s) = \Psi_1(\zeta_1)\frac{kT\zeta_1}{s^2+(kT\zeta_1)^2} + \frac{2}{\pi}\int_1^{\infty} \Psi(\zeta)\frac{kT\zeta}{s^2+(kT\zeta)^2}\,d\zeta \tag{4.11}$$

For real u the function $f(p) = p/(p^2+u^2)$ is analytic in the circle $|p-1| \leqslant 1$ and is represented in the form of the series

$$f(p) = \sum_{m=0}^{\infty} f_m(u)(p-1)^m \tag{4.12}$$

$$f_m(u) = \frac{1}{2}(-1)^m [(1+iu)^{-(m+1)} + (1-iu)^{-(m+1)}] \quad (m=0,1,2,\ldots) \tag{4.13}$$

It follows from (4.12) that

$$F_{km} = \frac{1}{kT}\left[\frac{\Psi_1(\zeta_1)}{\zeta_1} f_m\left(\frac{1}{kT\zeta_1}\right) + \frac{2}{\pi}\int_1^{\infty} \frac{\Psi(\zeta)}{\zeta} f_m\left(\frac{1}{kT\zeta}\right) d\zeta\right] \tag{4.14}$$

The expression in square brackets is bounded uniformly for all k and m. Hence, as well as from the asymptotic representation

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right] \tag{4.15}$$

the convergence of series (4.4) follows.

5. We will show that each of the system (4.8) is quasiregular and can be solved by

reduction.

We introduce the notation

$$G_{2n,2j,m} = \sum_{k=1}^{\infty} \frac{J_{2n}(kT)\,J_{2j}(kT)}{k^m} \tag{5.1}$$

Substituting the Sommerfeld integral representation for the Bessel functions

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix\sin\varphi - in\varphi}\, d\varphi$$

into (5.1), and carrying out the transformation, we obtain

$$G_{2n,2j,1} = \frac{(-1)^{n+j}}{2\pi^2} \int_{-1}^{1}\int_{-1}^{1} T_{2n}(\xi)\,T_{2j}(\eta) \times \ln\left|\frac{1}{\sin {}^1/{}_2T(\xi-\eta)}\right| \frac{d\xi}{\sqrt{1-\xi^2}}\,\frac{d\eta}{\sqrt{1-\eta^2}} \tag{5.2}$$

Representing the sine in the form of an infinite product and using the well-known spectral relationship

$$\frac{1}{\pi}\int_{-1}^{1} \frac{T_m(\eta)}{\sqrt{1-\eta^2}} \ln\left|\frac{1}{\xi-\eta}\right| d\eta = \mu_m T_m(\xi)$$

$$(\mu_0 = \ln 2,\ \mu_m = m^{-1},\ m > 0)$$

we will have

$$G_{2n,2j,1} = \frac{2(-1)^{n+j}\pi^2}{n}\,\delta_{nj} + 8(-1)^{n+j}\alpha_{2n,2j},\quad n \geqslant 1 \tag{5.3}$$

$$\alpha_{nj} = \sum_{k=1}^{\infty} \frac{T^{2k}\zeta(2k)}{(2\pi)^{2k}} \int_{-1}^{1}\int_{-1}^{1} (\xi-\eta)^{2k}\, T_n(\xi)\,T_j(\eta)\, \frac{d\xi}{\sqrt{1-\xi^2}}\,\frac{d\eta}{\sqrt{1-\eta^2}}$$

($\zeta(x)$ is the Riemann zeta function, and δ_{nj} is the Kronecker delta).

It follows from the properties of Chebyshev polynomials that

$$|\alpha_{2n,2j}| \leqslant \pi^2\zeta(2)\left(\frac{T}{\pi}\right)^{2(n+j)}\left[1-\left(\frac{T}{\pi}\right)^2\right]^{-1} \tag{5.4}$$

We note that since $l > 2$, then $T < \pi$.

Using the recursion relations for Bessel functions, we can show that

$$G_{2n,2j,3} = -\frac{T^2(-1)^{n+j}}{4nj}\left[\pi^2\left(\frac{\delta_{nj}-\delta_{n,j+1}}{2n-1} - \frac{\delta_{n+1,j}-\delta_{nj}}{2n+1}\right) + 2\left(\alpha_{2n-1,2j-1} + \alpha_{2n-1,2j+1} + \alpha_{2n+1,2j-1} + \alpha_{2n+1,2j+1}\right)\right]$$

The series

$$G'_{2n,2j,5}(\zeta) = \sum_{k=1}^{\infty} \frac{J_{2n}(kT)\,J_{2j}(kT)}{k^5}\left[1+\left(\frac{1}{kT\zeta}\right)^2\right]^{-1} \quad (n, j \geqslant 1) \tag{5.5}$$

satisfies the inequality

$$|G'_{2n,2j,5}(\zeta)| \leqslant \frac{C}{n^2j^2} \tag{5.6}$$

where C is a certain constant independent of n and j.

Substituting (4.14) into (4.4) we have

$$C_{2n,2j,m} = \frac{(-1)^m}{2}\left[\frac{\Psi_1(\zeta_1)}{\zeta_1} E_{2n,2j,m}(\zeta_1) + \frac{2}{\pi}\int_1^{\infty} \frac{\Psi(\zeta)}{\zeta} E_{2n,2j,m}(\zeta)\, d\zeta\right] \tag{5.7}$$

$$E_{2n,2j,m}(\zeta) = \frac{1}{T}\sum_{k=1}^{\infty} f_m\left(\frac{1}{kT\zeta}\right)\frac{J_{2n}(kT)\,J_{2j}(kT)}{k}$$

where the functions $f_m(u)$ are given by (4.13).

It can be seen that

$$E_{2n,2j,0}(\zeta) = \frac{1}{T}\left[G_{2n,2j,1} - \frac{1}{(T\zeta)^2}G_{2n,2j,3} + \frac{1}{(T\zeta)^4}G'_{2n,2j,5}(\zeta)\right] \tag{5.8}$$

It follows from (5.3)–(5.8) and (4.3) that for $n,\ j \geqslant 1$

$$B_{2n,2n,0} = \frac{C_1}{n} + O(1/n^2),\quad |B_{2n,2j,0}| \leqslant \frac{C_2}{n^2j^2} \quad (n \neq j) \tag{5.9}$$

where C_1, C_2 are certain constants independent of n and j.

It follows from (5.9) that

$$\lim_{n\to\infty}\frac{S_{2n}}{B_{2n,\,2n,\,0}}=0,\quad S_{2n}=\sum_{j=0,\;j\neq n}^{\infty}|B_{2n,\,2j,\,0}| \tag{5.10}$$

It can similarly be shown that the following relationship holds:

$$B_{2n,\,2j,\,m}=O\ (1/n) \tag{5.11}$$

Hence, and from (4.9) the boundedness of the right side in (4.8) results. Taking account of (5.10) we arrive at the conclusion that system (4.8) is quasiregular.

Because of the successive solution of the truncated systems for $m = 0, 1, 2, \ldots$ we obtain the coefficients $A_{2j,m}$ ($j = 0, 1, 2, \ldots, N$; $m = 0, 1, 2, \ldots$) of the expansion (4.6). Carrying out the reverse substitution $p = 1/s$ in (4.6), and going over to the originals, we obtain ($L_m(t)$ is the Laguerre polynomial)

$$A_{2j}(t)=\sum_{m=0}^{\infty}A_{2j,\,m}(-1)^m L_m(t) \tag{5.12}$$

6. We will now estimate the convergence of series (5.12). We find the solution of the truncated system (3.6) by Cramer's rule

$$\bar{A}_{2j}(s)=\Delta_j(s)/\Delta(s)\qquad (j=0,1,2,\ldots,N) \tag{6.1}$$

In this formula $\Delta(s)$ is the system determinant whose order is $N+1$, while $\Delta_j(s)$ is the determinant of the matrix obtained by replacing a column of the matrix of system (3.6) by a column of free terms.

Let $N = 0$. Then

$$A_0(s)=\frac{g}{\pi a}\frac{P_0(s)}{s^3B_{00}(s)} \tag{6.2}$$

The deduction can be made from (3.7) that the function $B_{00}(s)$ is analytic in the right half-plane. It follows from physical considerations (this can also be shown by methods of the theory of analytic functions) that the zeros of $B_{00}(s)$ are in the left half-plane. Since the line $s = 1/2 + iy$ $(-\infty < y < \infty)$ under the transformation $s = 1/p$ transforms into the circle $|p - 1| = 1$, the function $n/B_{00}(1/p)$ is infinitely differentiable in this circle with the point $p = 0$ excluded.

Consider the function

$$H_\zeta(s)=\sum_{k=1}^{\infty}[s^2+(kT\zeta)^2]^{-1}$$

By using the formula (5.1.25,4) in /8/, we obtain

$$H_\zeta(s)=\frac{1}{2}\left(-\frac{1}{s^2}+\frac{\pi}{sT\zeta}\operatorname{cth}\frac{\pi s}{T\zeta}\right) \tag{6.3}$$

It can be seen that on the line $s = {}^1/_2 + iy$ $(-\infty < y < \infty)$ for fixed ζ

$$H_\zeta(s)=O\ (1/s) \tag{6.4}$$

We now examine the integral

$$I(s)=\int_1^{\infty}H_\zeta(s)\,d\zeta.$$

Substituting $z = s/\zeta$ we obtain

$$I(s)=\frac{1}{2s}\int_0^{s}\left(-1+\frac{\pi z}{T}\operatorname{cth}\frac{\pi z}{T}\right)\frac{dz}{z^2}$$

Replacing integration along the segment $[0, s]$ by integrating along two segments $[0, {}^1/_2]$ and $[{}^1/_2, s]$, for $s = {}^1/_2 + iy$ we will have

$$I(s)\sim\frac{\pi}{2Ts}\int_0^{y}\operatorname{cth}\frac{\pi({}^1/_2+i\eta)}{T}\,\frac{d\eta}{{}^1/_2+i\eta}\sim-\frac{C\pi\ln|y|}{4Ty} \tag{6.5}$$

$$C=\frac{1}{2T}\int_0^{T}\operatorname{Im}\operatorname{cth}\frac{\pi({}^1/_2+i\eta)}{T}\,d\eta$$

The following representation results from (3.7) and (4.11)

$$B_{00}(s)=\frac{\beta}{s}+\frac{g}{s^2}+2\Psi_1(\zeta_1)\zeta_1\sum_{k=1}^{\infty}\frac{I_0^2(kT)\,kT}{s^2+(kT\zeta_1)^2}+\frac{4}{\pi}\int_1^{\infty}\Psi(\zeta)\,\zeta\sum_{k=1}^{\infty}\frac{J_0^2(kT)\,kT}{s^2+(kT\zeta)^2}\,d\zeta \tag{6.6}$$

Using the asymptotic (4.15) as well as (6.4) and (6.5), we obtain

$$\frac{1}{sB_{00}(s)}=\frac{\pi Ti}{C\beta\ln|y|}+O\left(\frac{1}{y}\right)\quad (s={}^1/_2+iy,\ y\to\infty) \tag{6.7}$$

As already remarked above, under the transformation $s = 1/p$ the line $s = {}^1/_2 + iy$ transforms into the unit circle $p = 1 + e^{i\varphi}$ $(|\varphi| < \pi)$. Here $y = -{}^1/_2 \operatorname{tg} \varphi/2$. Consequently, the function $p/B_{00}(1/p)$, considered as a function of the parameter φ in the segment $(-\pi, \pi)$ has the following asymptotic representation in the neighbourhood of the points $\varphi = \pm\pi$:

$$\frac{p}{B_{00}(1/p)} = O\left(\frac{1}{\ln|\varphi - \pi|}\right) \quad (p = 1 + e^{i\varphi}) \tag{6.8}$$

It can be seen that a function continuously differentiable in the segment $(-\pi, \pi)$ and satisfying the asymptotic estimate (6.8) in the neighbourhood of the points $\varphi = \pm\pi$ is expanded in a Fourier series in $(-\pi, \pi)$ whose coefficients have the asymptotic behaviour presented below

$$\frac{p}{B_{00}(1/p)} = \sum_{k=0}^{\infty} a_k e^{ik\varphi} \, (p = 1 + e^{i\varphi}), \quad a_k = O(1/k) \tag{6.9}$$

On the boundary of the unit circle of convergence, the power series transforms into a Fourier series, hence (6.9) means that the function $p/B_{00}(1/p)$ is expanded in a circle $|p - 1| < 1$ in the power series

$$\frac{p}{B_{00}(1/p)} = \sum_{k=0}^{\infty} a_k (p-1)^k \tag{6.10}$$

Under the transformation $s = 1/p$ formula (6.2) transforms into the following expression:

$$\bar{A}_0\left(\frac{1}{p}\right) = \frac{g}{\pi a} \frac{\bar{P}_0(1/p)\, p^2}{B_{00}(1/p)} \tag{6.11}$$

In many important cases the function $\bar{P}_0(s)$ is analytic in the half-plane $\operatorname{Re} s \geqslant {}^1/_2$ and has the following asymptotic form there:

$$\bar{P}_0(s) = O(1/s) \tag{6.12}$$

As follows from (6.10) and (6.12), the left side of (6.11), meaning also (4.6), is expanded in a power series in $p - 1$ for $j = 0$, whose coefficients satisfy the asymptotic estimate

$$A_{0,m} = O(1/m^2) \tag{6.13}$$

Analogous, but somewhat more awkward, computations can be performed for $N > 0$ also. Consequently, we have shown that the coefficients of the series (4.6) satisfy the asymptotic form (6.13) even for $j > 0$.

It follows from formula (8.978.3) of /9/ that series (5.12) converges uniformly in any segment of the form $[t_0, t_1]$ $(0 < t_0 < t_1 < \infty)$.

7. Equation (3.2) is investigated and solved in exactly the same way. In this case the contact stresses are determined by the series

$$p_{01}(x, t) = \sum_{j=0}^{\infty} A_{2j+1}(t) \frac{T_{2j+1}(x)}{\sqrt{1 - x^2}} \tag{7.1}$$

whose coefficients $A_{2j+1}(t)$ $(j = 0, 1, 2, \ldots)$ are found by using the algorithm described above.

Note that the solution of (3.1) is symmetric relative to reflection in a plane perpendicular to the sketch and passing through the Oy axis. Hence and from the periodicity of the problem it follows that the state of stress and strain of an elastic half-plane is invariant to reflection in planes parallel to the plane mentioned and intersecting the Ox axis at the points $x_k = kl/2$ $(k = 0, \pm1, \pm2, \ldots)$. From this fact it follows that the solution of the equation agrees with the solution of the contact problem for an elastic half-strip compressed between two parallel, absolutely smooth directrices (Fig.1). The solution of (3.2) is skew-symmetric about the Oy axis and agrees with the solution of the contact problem for a half-strip, on whose parallel edges are superposed constraints hindering the vertical displacements.

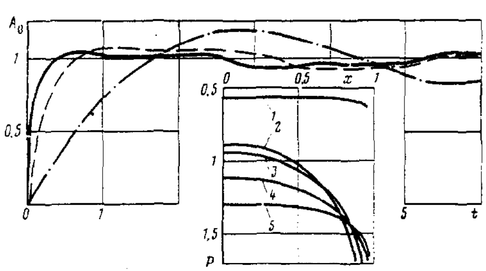

Fig.2

Example. A vertical constant force $P_0(t) = \pi a H(t)$ ($H(t)$ is the Heaviside unit function) is applied suddenly to the middle of a stamp lying on the endface of a half-strip. An already remarked above, (3.1) corresponds to this problem. The contact stresses are sought in the form of the series (3.4). Realizing the algorithm represented by (4.8) and (4.9), in which we must put

$$Q_0 = Q_1 = \pi a, \quad Q_m = 0 \ (m \geqslant 2)$$

we obtain the coefficients $A_{2j,m}$ $(j = 0, 1, 2, \ldots, N;\ m = 0, 1, 2, \ldots, M)$ of the expansion (5.12) of the function $A_{2j}(t)$ $(j = 0, 1, 2, \ldots, N)$ in Laguerre polynomials. Here N is determined by the order of the truncated system (4.8), and M is the number of terms retained in series (5.12).

Graphs of the function $A_0(t)$ are represented in Fig.2 for the values $g = 1, 5, 10$ (the continuous, dashed, and dash-dot lines, respectively) for $T = \pi/2$, $\beta = 0.535$, $\zeta_1 = 0.927$ and diagrams of the contact stresses are displayed for different values of t at $g = 5$. Curve *1* corresponds to the value $t = 0.1$, curve *2* to the value $t = 2.6$; the stress diagrams for $t = 0.6$ and 3.6 (curve *3*), for $t = 1.1$ and 3.1 (curve *4*), and also for $t = 1.6$ and 2.1 (curve *5*) agree practically in pairs.

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THERMOPHORESIS AND THE INTERACTION OF UNIFORMLY HEATED SPHERICAL PARTICLES IN A GAS*

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The thermophoresis of a uniformly heated spherical particle caused by the action of Branett temperature stresses is investigated, and the thermophoretic force is calculated for arbitrary temperature drops between the particle and the gas. An analogous problem was considered earlier /1/ in the linear approximation of a small temperature drop.

The results obtained are used to estimate the nature and interaction force of widely spaced particles. It is shown that the gas motion caused by the temperature stresses can result in displacement of the system of differently heated particles.

We consider a uniformly heated (cooled) spherical particle in a gas at rest at infinity whose temperature varies weakly along the x axis. The gas is regarded as a continuous medium. The temperature stresses evoke a pressure redistribution and gas motion around the particles /2/, which will result in the appearance of a thermophoretic force acting on the particle.

We introduce dimensionless coordinates, temperature, density, viscosity, thermal conductivity, velocity, pressure, and force as follows:

$$a(x\ y\ z),\quad T_\infty T,\quad \rho_\infty\rho\quad \mu_\infty\mu,\quad \lambda_\infty\lambda$$

$$\frac{\mu_\infty}{\rho_\infty a}\mathbf{v},\quad \frac{P}{P_\infty} = 1 + \left[\frac{\mu_\infty}{\rho_\infty a}(RT_\infty)^{-1/2}\right]p,\quad \frac{{\mu_\infty}^2}{\rho_\infty}\mathbf{F}$$

Here a is the radius of the sphere; when there is no temperature gradient at infinity, the subscript ∞ is ascribed to the appropriate gas parameters far from the sphere. The dimensionless continuity, energy, and momentum equations describing the flow around the particle /2/, and the boundary conditions can be written in the following form:

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